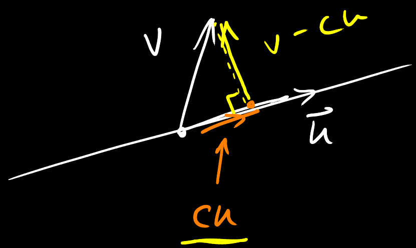
Orthogonality

<u>Recall:</u> Vectors u, v in $\mathbb{R}^n$ are <u>orthogonal</u> (or <u>perpendicular</u>) when $u \cdot v = 0$.

$\Big($<u>idea:</u> $u \cdot v = 0 \Rightarrow 0 = u \cdot v = |u||v|\cos(\theta)$
so provided $u \neq \vec{0} \neq v$, we see $\cos(\theta) = 0 \Rightarrow \theta = \frac{\pi}{2}\Big)$

<u>Q:</u> Can we project vectors orthogonally?
i.e. Can we measure "<u>how far v tends in direction of u</u>"?

<u>A:</u> Yes!



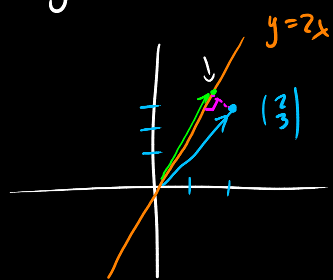
<u>Derivation:</u> Given two vectors $u, v \in \mathbb{R}^n$ w/ $\boxed{u \neq \vec{0}.}$ We seek a vector cu w/ $v - cu$ is orthogonal to u.

i.e. $u \cdot (v - cu) = 0$ so $u \cdot v - c(u \cdot u) = 0$, which yields $c(u \cdot u) = u \cdot v$, so $c = \frac{u \cdot v}{u \cdot u}$ noting $u \cdot u \neq 0$.

Hence $\boxed{\operatorname{proj}_{\operatorname{span}(u)}(v) = cu = \left(\frac{u \cdot v}{u \cdot u}\right)u}$ ☺.

↑ projection of v onto the span of u.

<u>Ex:</u> Compute the projection of $\begin{pmatrix}2\\3\end{pmatrix}$ onto the line $y = 2x$ in $\mathbb{R}^2$



<u>Sol:</u> We choose a vector in the direction of the line $y = 2x$:

$$\ell = \left\{\begin{pmatrix}x\\y\end{pmatrix} : y = 2x\right\} = \left\{\begin{pmatrix}x\\2x\end{pmatrix} : x \in \mathbb{R}\right\} = \operatorname{span}\left\{\begin{pmatrix}1\\2\end{pmatrix}\right\}.$$

$$\therefore \operatorname{proj}_\ell\begin{pmatrix}2\\3\end{pmatrix} = \frac{\begin{pmatrix}2\\3\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}}{\begin{pmatrix}1\\2\end{pmatrix}\cdot\begin{pmatrix}1\\2\end{pmatrix}}\begin{pmatrix}1\\2\end{pmatrix} = \frac{2+6}{1+2^2}\begin{pmatrix}1\\2\end{pmatrix} = \frac{8}{5}\begin{pmatrix}1\\2\end{pmatrix}. \quad \boxtimes$$

<u>Ex:</u> Compute the orthogonal projection of $\begin{pmatrix}1\\2\\1\\3\end{pmatrix}$ onto $\operatorname{span}\left\{\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}\right\}$.

<u>Sol:</u> $v = \begin{pmatrix}1\\2\\1\\3\end{pmatrix}$, $u = \begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$ so $\operatorname{proj}_{\operatorname{span}(u)}(v) = \frac{u \cdot v}{u \cdot u}u$

$u \cdot v = -1 + 2 - 1 + 3 = 3$

$u \cdot u = (-1)^2 + 1^2 + (-1)^2 + 1^2 = 4$ and thus $\text{proj}_{\text{span}(u)}(v) = \frac{3}{4}u = \frac{3}{4}\begin{pmatrix}-1\\1\\-1\\1\end{pmatrix}$. $\boxtimes$

<u>Defn</u>: A collection $\{v_1, v_2, \ldots, v_n\}$ is <u>pairwise orthogonal</u> (aka <u>mutually orthogonal</u>) when every pair of distinct vectors v_i, v_j is an orthogonal pair. I.e. for all $1 \le i < j \le n$ we have $v_i \cdot v_j = 0$.

<u>Ex</u>: $\mathcal{E}_n \leftarrow$ the standard basis on $\mathbb{R}^n$ is a pairwise orthogonal collection.

$$e_i \cdot e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

<u>Ex</u>: $\left\{\begin{pmatrix}4\\2\end{pmatrix}, \begin{pmatrix}1\\3\end{pmatrix}\right\}$ are <u>not</u> mutually orthogonal.

$$\begin{pmatrix}4\\2\end{pmatrix} \cdot \begin{pmatrix}1\\3\end{pmatrix} = 4 + 6 = 10 \neq 0 \ldots$$

<u>Q</u>: Can we modify the collection to build a mutually orthogonal one?

<u>Prop</u>: If $S = \{v_1, v_2, \ldots, v_n\}$ is a collection of pairwise orthogonal <u>nonzero</u> vectors, then S is lin. indep.

<u>pf</u>: Assume S is a collection of pairwise orthogonal nonzero vectors, and suppose $c_1 v_1 + c_2 v_2 + \cdots + c_n v_n = \vec{0}$. Now $v_i \cdot v_j = 0$ when $i \neq j$, and nonzero when $i = j$. Hence:

$$v_i \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = v_i \cdot \vec{0} = 0$$

OTOH: $v_i \cdot (c_1 v_1 + c_2 v_2 + \cdots + c_n v_n) = c_1 (v_i \cdot v_1) + c_2 (v_i \cdot v_2) + \cdots + c_n (v_i \cdot v_n)$

$$= c_1 0 + c_2 0 + \cdots + c_i (v_i \cdot v_i) + \cdots + c_i 0$$

$$= 0 + 0 + \cdots + c_i (v_i \cdot v_i) + \cdots + 0$$

$$= c_i (v_i \cdot v_i).$$

So $0 = c_i (v_i \cdot v_i)$, and $v_i \cdot v_i \neq 0$ because $v_i \neq \vec{0}$; thus $c_i = 0$.

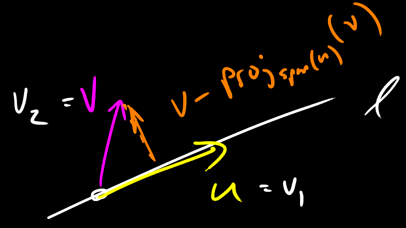
Hence $c_i = 0$ for all $1 \le i \le n$, and we see S is lin. ind. $\boxtimes$

<u>Point</u>: Mutually orthogonal nonzero vectors are linearly independent :-).

Cor: If S is a collection of n mutually orthogonal vectors in $\mathbb{R}^n$, then S is a basis for $\mathbb{R}^n$. ☺

Returning to the example from before: $S = \left\{ v_1 = \begin{pmatrix} 4 \\ 2 \end{pmatrix}, v_2 = \begin{pmatrix} 1 \\ 3 \end{pmatrix} \right\}$.

Goal: Build a collection $\hat{S}$ of vectors based on S which is a mutually orthogonal collection...



Start Building $\hat{S}$: $\hat{S}_1 = \{ u_1 = v_1 \}$

Let $u_2 = v_2 - \mathrm{proj}_{\mathrm{span}(u_1)}(v_2)$

$$= \begin{pmatrix} 1 \\ 3 \end{pmatrix} - \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

$\mathrm{proj}_{\mathrm{span}(u_1)}\begin{pmatrix} 1 \\ 3 \end{pmatrix}$

$$= \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1 = \frac{10}{4^2+2^2}\begin{pmatrix} 4 \\ 2 \end{pmatrix}$$

$$= \frac{10}{20}\begin{pmatrix} 4 \\ 2 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Let $\hat{S}_2 = \{ u_1, u_2 \}$. Claim: $\hat{S}$ is mutually orth. coll.

Check: $u_1 \cdot u_2 = \begin{pmatrix} 4 \\ 2 \end{pmatrix} \cdot \begin{pmatrix} -1 \\ 2 \end{pmatrix} = 4 \cdot (-1) + 2 \cdot 2 = 0$ ☑

Point: Projections allow us to build mutually orthogonal collections of vectors from arbitrary lin. indep collections in $\mathbb{R}^n$.

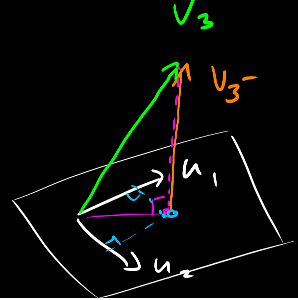
Q: How important was the fact we had only two vectors?

Ex: Consider the basis $S = \left\{ \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 3 \end{pmatrix} \right\}$ for $\mathbb{R}^3$.
$\qquad\qquad v_1 \qquad v_2 \qquad v_3$

* $u_1 = v_1$
* $u_2 = v_2 - \mathrm{proj}_{\mathrm{span}(u_1)}(v_2) = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \dfrac{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix}}{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 0 \end{pmatrix} - \dfrac{2}{3}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} -\frac{2}{3} \\ \frac{4}{3} \\ -\frac{2}{3} \end{pmatrix}$

$$= \frac{2}{3}\begin{pmatrix} -1 \\ 2 \\ -1 \end{pmatrix}$$

NB: For a basis of orthogonal vectors, the representation of every $w \in \mathbb{R}^n$ w.r.t. the orthogonal basis is determined by the dot product w/ each vector of the basis...



$$u_3 = v_3 - \underline{\mathrm{proj}_{\mathrm{span}(u_2)}(v_3)} - \underline{\mathrm{proj}_{\mathrm{span}(u_1)}(v_3)}$$

$$v_3 = \begin{pmatrix}1\\0\\3\end{pmatrix}$$

$$u_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, \quad u_2 = \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}$$

$$= v_3 - \frac{u_2 \cdot v_3}{u_2 \cdot u_2} u_2 - \frac{u_1 \cdot v_3}{u_1 \cdot u_1} u_1$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} - \frac{\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}}{\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}} \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\0\\3\end{pmatrix}}{\begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}1\\1\\1\end{pmatrix}}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} - \left(\frac{-1-3}{1+4+1}\right)\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}1\\0\\3\end{pmatrix} + \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} - \frac{4}{3}\begin{pmatrix}1\\1\\1\end{pmatrix} = \frac{1}{3}\begin{pmatrix}3-2-4\\0+4-4\\9-2-4\end{pmatrix} = \frac{1}{3}\begin{pmatrix}-3\\0\\3\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}.$$

<u>Check:</u> $\hat{S} = \{u_1, u_2, u_3\}$ is a collection of mutually orthogonal vectors.

$$u_1 \cdot u_2 = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix} = \frac{2}{3}(-1+2-1) = 0$$

$$u_1 \cdot u_3 = \begin{pmatrix}1\\1\\1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = -1+0+1 = 0$$

$$u_2 \cdot u_3 = \frac{2}{3}\begin{pmatrix}-1\\2\\-1\end{pmatrix}\cdot\begin{pmatrix}-1\\0\\1\end{pmatrix} = \frac{2}{3}(1+0-1) = 0$$

$\boxtimes$

<u>Prop (Gram-Schmidt Process):</u> Let $v_1, v_2, \dots, v_k$ be a sequence of linearly independent vectors in $\mathbb{R}^n$. The sequence

$$u_1 = v_1$$

$$u_2 = v_2 - \frac{u_1 \cdot v_2}{u_1 \cdot u_1} u_1$$

$$\vdots$$

$$u_k = v_k - \frac{u_{k-1} \cdot v_k}{u_{k-1} \cdot u_{k-1}} u_{k-1} - \frac{u_{k-2} \cdot v_k}{u_{k-2} \cdot u_{k-2}} u_{k-2} - \dots - \frac{u_1 \cdot v_k}{u_1 \cdot u_1} u_1$$

is a collection of mutually orthogonal vectors. $\boxtimes$